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A q-Analogue of Graf's Addition Formula for the Hahn–Exton q-Bessel Function

H. T. KOELINK*

Departement Wiskunde, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Heverlee, Belgium

AND

R. F. SWARTTOUW†

Faculteit Wiskunde en Informatica, Vrije Universiteit, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

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An addition and product formula for the Hahn–Exton q-Bessel function, previously obtained by use of a quantum group theoretic interpretation, are proved analytically. A (formal) limit transition to the Graf addition formula and corresponding product formula for the Bessel function is given. © 1995 Academic Press, Inc.

1. INTRODUCTION

A classical result for the Bessel function

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)} \tag{1.1}$$

is the addition formula

$$J_\nu(\sqrt{x^2+y^2-2xy\cos\psi}) \left(\frac{x-ye^{-i\psi}}{x-ye^{i\psi}}\right)^{\nu/2} = \sum_{m=-\infty}^{\infty} J_{\nu+m}(x)\, J_m(y)\, e^{im\psi}, \tag{1.2}$$

$|ye^{\pm i\psi}| < |x|$, due to Graf (1893), cf. [19, Sect. 11.3(1)], for general ν and

* Supported by a NATO Science Fellowship of the Netherlands Organization for Scientific Research (NWO). E-mail address: Erik.Koelink@wis.KULeuven.ac.be.

† E-mail address: rene@cs.vu.nl.

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due to Neumann (1867) for $\nu=0$, cf. [19, Sect. 11.2(1)]. In the case $\nu\in\mathbf{Z}$; the conditions on x, y in (1.2) can be removed.

Several q-analogues of the Bessel function (1.1) have been studied. The oldest q-analogues have been introduced by Jackson in 1903–1905, cf. the references in Ismail [4]. The q-Bessel function studied in this note is the so-called Hahn–Exton q-Bessel function which has been introduced by Hahn (1949) for a special case and by Exton (1978) in full generality; cf. references in Koornwinder and Swarttouw [13].

The Hahn–Exton q-Bessel function is defined by

$$J_\alpha(z;q)=z^\alpha\frac{(q^{\alpha+1};q)_\infty}{(q;q)_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ q^{\alpha+1}\end{matrix};q,qz^2\right). \tag{1.3}$$

Here $q\in(0,1)$, $(a;q)_0=1$, $(a;q)_k=\prod_{i=0}^{k-1}(1-aq^i)$ for $k\in\mathbf{N}$, $(a;q)_\infty=\lim_{k\to\infty}(a;q)_k$, and the q-hypergeometric function is defined by

$$\begin{aligned}&{}_r\varphi_s\left(\begin{matrix}a_1,a_2,...,a_r\\ b_1,...,b_s\end{matrix};q,z\right)\\ &\quad=\sum_{k=0}^{\infty}\frac{(a_1;q)_k\,(a_2;q)_k\cdots(a_r;q)_k}{(q;q)_k\,(b_1;q)_k\cdots(b_s;q)_k}z^k((-1)^k\,q^{(1/2)k(k-1)})^{(s-r+1)}.\end{aligned}$$

The notation for q-shifted factorials and q-hypergeometric series is taken from the book by Gasper and Rahman [3] to which the reader is referred for more information on this subject.

The goal of this note is to prove the formula

$$\begin{aligned}&J_\nu(Rq^{(1/2)(y+z+\nu)};q)\,J_{x-\nu}(q^{(1/2)z};q)\\ &\quad=\sum_{k=-\infty}^{\infty}J_k(Rq^{(1/2)(x+y+k)};q)\,J_{\nu+k}(Rq^{(1/2)(y+k+\nu)};q)\,J_x(q^{(1/2)(z-k)};q).\end{aligned} \tag{1.4}$$

This formula is valid for $z\in\mathbf{Z}$, $R, x, y, \nu\in\mathbf{C}$ satisfying $q^{1+\Re(x)+\Re(y)}\,|R|^2<1$, $\Re(x)>-1$, and $R\neq0$.

The formula (1.4) has originally been derived for $\nu, x, y\in\mathbf{Z}$, $R>0$ by Koelink using the interpretation of the Hahn–Exton q-Bessel function as matrix elements of irreducible unitary representations of the quantum group of plane motions, cf. [7, Sect. 6]. This quantum group theoretic interpretation of the Hahn–Exton q-Bessel function is due to Vaksman and Korogodskiĭ [16]. The quantum group theoretic derivation of (1.4) in [7] is modelled on the group theoretic derivation of Graf's addition formula (1.2) as presented by Vilenkin and Klimyk [18, Sect. 4.1.4(2)], so we call (1.4) a q-analogue of Graf's addition formula for the Hahn–Exton q-Bessel function.

In Section 4 we present a (formal) limit transition of (1.4) to (1.2) as q tends to 1. The approach employed is based on a theorem presented by Van Assche and Koornwinder [17] which has been used to show that the addition formula for the little q-Legendre polynomial [12] tends to the addition formula for the Legendre polynomial. The case $\nu=0$, $R=q^{-1/2}$ of the addition formula (1.4) can be obtained by taking a (formal) limit in Koornwinder's [12] addition formula for the little q-Legendre polynomials using the limit transition of the little q-Jacobi polynomials to the Hahn–Exton q-Bessel function, cf. [13, Prop. A.1].

Kalnins *et al.* [6] have given another derivation of the addition formula (1.4). They consider representations of the Lie algebra of the group of orientation and distance preserving motions of the plane. Instead of exponentiating the representations using the exponential function, they use a q-analogue of the exponential function. In a particular case the matrix elements can be expressed in terms of the Hahn–Exton q-Bessel functions. They give a decomposition of the tensor product (not the standard tensor product, but one closely related to quantum groups) of two representations and the corresponding Clebsch–Gordan coefficients are in terms of the Hahn–Exton q-Bessel functions. Comparison of matrix coefficients yields the addition formula (1.4).

More results on Graf-type addition formulas for the Hahn–Exton q-Bessel function, the Jackson q-Bessel function, and other q-analogues of the Bessel function can be found in [1, 2, 5–9, 13].

We start the proof of the q-analogue of Graf's addition formula (1.4) for the Hahn–Exton q-Bessel function by proving the product formula

$$(-1)^m q^{-(1/2)m} J_m(Rq^{(1/2)(x+y)}; q)\, J_{\nu-m}(Rq^{(1/2)(y+\nu-m)}; q)$$
$$= \sum_{z=-\infty}^{\infty} q^z J_x(q^{(1/2)(m+z)}; q)\, J_{x-\nu}(q^{(1/2)z}; q)\, J_\nu(Rq^{(1/2)(y+\nu+z)}; q) \qquad (1.5)$$

valid for $m \in \mathbf{Z}$, $R, x, y, \nu \in \mathbf{C}$ satisfying $\Re(x) > -1$, $q^{1+\Re(x)+\Re(y)}\,|R|^2 < 1$, and $R \neq 0$. This is done in Section 2. The proof of the addition formula (1.4) is a direct consequence of (1.5) as is shown in Section 3. The product formula in the case $\nu=m=0$, $R=q^{-1/2}$ is mentioned (without proof) by Vaksman and Korogodskiĭ [16, p. 177].

In Section 5 we will show that (1.5) is a q-analogue of the product formula for Bessel functions,

$$J_{\nu+m}(x)\, J_m(y) = \frac{1}{2\pi} \int_0^{2\pi} J_\nu(\sqrt{x^2+y^2-2xy\cos\psi}) \left(\frac{x-ye^{-i\psi}}{x-ye^{i\psi}}\right)^{\nu/2} e^{-im\psi}\, d\psi. \qquad (1.6)$$

The product formula is a direct consequence of Graf's addition formula (1.2).

2. Proof of the Product Formula

In this section we present an analytic proof of the product formula (1.5). The proof uses two known formulas for the Hahn–Exton q-Bessel function previously obtained by Koornwinder and Swarttouw [13] and Swarttouw [14].

The proof starts with the following formula, valid for $|sxy| < 1$ and $m \in \mathbf{Z}$;

$$y^m \frac{(s^{-1}xy^{-1}; q)_\infty (q^{m+1}; q)_\infty}{(q^m s^{-1}xy^{-1}; q)_\infty (q; q)_\infty} {}_2\varphi_1\left(\begin{matrix} q^m s^{-1}xy^{-1}, s^{-1}yx^{-1} \\ q^{m+1} \end{matrix}; q, sxy\right)$$

$$= \sum_{z=-\infty}^{\infty} s^z y^{m+z} \frac{(y^2; q)_\infty}{(q; q)_\infty} {}_1\varphi_1\left(\begin{matrix} 0 \\ y^2 \end{matrix}; q, q^{m+z+1}\right)$$

$$\times x^z \frac{(x^2; q)_\infty}{(q; q)_\infty} {}_1\varphi_1\left(\begin{matrix} 0 \\ x^2 \end{matrix}; q, q^{z+1}\right), \tag{2.1}$$

cf. [13, (4.5), (2.3)].

In (2.1) we take $x = q^{(1/2)(\chi - \nu + 1)}$, $y = q^{(1/2)(\chi + 1)}$ and $s = q^{(1/2)\nu + n}$ for $n \in \mathbf{Z}_+$ to get

$$q^{(1/2)m\chi} \frac{(q^{-\nu-n}; q)_\infty (q^{m+1}; q)_\infty}{(q^{m-\nu-n}; q)_\infty (q; q)_\infty} {}_2\varphi_1\left(\begin{matrix} q^{m-\nu-n}, q^{-n} \\ q^{m+1} \end{matrix}; q, q^{\chi+n+1}\right)$$

$$= \sum_{z=-\infty}^{\infty} q^{z(n+1+(1/2)\nu)} J_\chi(q^{(1/2)(m+z)}; q)\, J_{\chi-\nu}(q^{(1/2)z}; q) \tag{2.2}$$

valid for $m \in \mathbf{Z}$, $\nu \in \mathbf{C}$, $\Re(\chi) > -1$, and all $n \in \mathbf{Z}_+$. In (2.2) we use the notation (1.3).

Multiply both sides of (2.2) by

$$\frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} q^{(1/2)\nu(\nu+\gamma)} \frac{(-1)^n q^{(1/2)n(n+1)}}{(q^{\nu+1}; q)_n (q; q)_n} q^{n(\gamma+\nu)} R^{\nu+2n}$$

and sum from $n = 0$ to ∞. After interchanging the summations over $z \in \mathbf{Z}$ and $n \in \mathbf{Z}_+$, which is justified for $\Re(\chi) > -1$ and $q^{1+\Re(\chi)+\Re(\gamma)} |R|^2 < 1$, cf. Proposition A.1, we find

$$\sum_{z=-\infty}^{\infty} q^z J_x(q^{(1/2)(m+z)};q)\, J_{x-\nu}(q^{(1/2)z};q)\, J_\nu(Rq^{(1/2)(y+\nu+z)};q)$$

$$= q^{(1/2)mx} \frac{(q^{m+1};q)_\infty (q^{\nu+1};q)_\infty}{(q;q)_\infty (q;q)_\infty} q^{(1/2)\nu(\nu+y)} R^\nu$$

$$\times \sum_{n=0}^{\infty} \frac{(q^{-\nu-n};q)_\infty}{(q^{m-\nu-n};q)_\infty} \frac{(-1)^n q^{(1/2)n(n+1)}}{(q^{\nu+1};q)_n (q;q)_n}$$

$$\times q^{n(y+\nu)} R^{2n} {}_2\varphi_1\left(\begin{matrix} q^{m-\nu-n}, q^{-n} \\ q^{m+1} \end{matrix}; q, q^{x+n+1}\right). \tag{2.3}$$

We use for $m \in \mathbf{Z}$, $n \in \mathbf{Z}_+$,

$$\frac{(q^{\nu+1};q)_\infty (q^{-\nu-n};q)_\infty}{(q^{\nu+1};q)_n (q^{m-\nu-n};q)_\infty} = (-1)^m q^{(1/2)m(m-1)-m(\nu+n)} \frac{(q^{\nu-m+1};q)_\infty}{(q^{\nu-m+1};q)_n}$$

to see that the right-hand side of (2.3) can be rewritten as

$$(-1)^m q^{-(1/2)m} \frac{(q^{m+1};q)_\infty (q^{\nu-m+1};q)_\infty}{(q;q)_\infty (q;q)_\infty} q^{(1/2)(\nu-m)(\nu-m+y)+(1/2)m(x+y)} R^\nu$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n q^{(1/2)n(n+1)}}{(q^{\nu-m+1};q)_n (q;q)_n} q^{n(y+\nu-m)} R^{2n} {}_2\varphi_1\left(\begin{matrix} q^{m-\nu-n}, q^{-n} \\ q^{m+1} \end{matrix}; q, q^{x+n+1}\right). \tag{2.4}$$

Now we apply the product formula for the Hahn–Exton q-Bessel function, cf. [14, (3.1)], which is valid for $a, b, x, \mu, \nu \in \mathbf{C}$ provided $abx \neq 0$. Explicitly,

$$J_\nu(ax;q)\, J_\mu(bx;q) = \frac{(q^{\nu+1};q)_\infty (q^{\mu+1};q)_\infty}{(q;q)_\infty (q;q)_\infty} a^\nu b^\mu x^{\nu+\mu}$$

$$\times \sum_{n=0}^{\infty} \frac{(-1)^n (bx)^{2n} q^{(1/2)n(n+1)}}{(q^{\mu+1};q)_n (q;q)_n}$$

$$\times {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-n-\mu} \\ q^{\nu+1} \end{matrix}; q, q^{\mu+n+1}\frac{a^2}{b^2}\right). \tag{2.5}$$

From (2.5) we see that (2.4) equals

$$(-1)^m q^{-(1/2)m} J_m(Rq^{(1/2)(x+y)};q)\, J_{\nu-m}(Rq^{(1/2)(y+\nu-m)};q),$$

which proves the product formula (1.5).

3. Proof of the Addition Formula

The proof of the addition formula (1.4) uses the orthogonality relations

$$\sum_{m=-\infty}^{\infty} q^{m+z} J_x(q^{(1/2)(z+m)}; q)\, J_x(q^{(1/2)(l+m)}; q) = \delta_{z,l}, \tag{3.1}$$

for $z, l \in \mathbf{Z}$, $\Re(x) > -1$; cf. [13, (2.11)]. In [13, Sect. 3] it is shown that (3.1) can be viewed as a q-analogue of the Hankel transform (or the Fourier–Bessel integral).

Multiply both sides of the product formula (1.5) by $q^m J_x(q^{(1/2)(l+m)}; q)$ for $l \in \mathbf{Z}$ and sum over m from $-\infty$ to ∞. We interchange the summations over $m \in \mathbf{Z}$ and $z \in \mathbf{Z}$, which is allowed for $\Re(x) > -1$, $q^{1+\Re(x)+\Re(y)} |R|^2 < 1$; cf. Proposition A.2. An application of the orthogonality relations (3.1) and replacing m by $-m$ yields

$$\begin{aligned} &J_\nu(Rq^{(1/2)(y+l+\nu)}; q)\, J_{x-\nu}(q^{(1/2)l}; q) \\ &\quad = \sum_{m=-\infty}^{\infty} (-1)^m q^{-(1/2)m} J_{-m}(Rq^{(1/2)(x+y)}; q) \\ &\qquad \times J_{\nu+m}(Rq^{(1/2)(y+m+\nu)}; q)\, J_x(q^{(1/2)(l-m)}; q). \end{aligned}$$

Since $J_{-n}(z; q) = (-1)^n q^{(1/2)n} J_n(zq^{(1/2)n}; q)$, $n \in \mathbf{Z}$ (cf. [13, (2.6)]), the addition formula (1.4) is proved.

4. The Limit Case $q \uparrow 1$ of the Addition Formula

In this section we present a (formal) limit transition of the q-analogue of the addition formula (1.4) for the Hahn–Exton q-Bessel function to Graf's addition formula (1.2) for the Bessel function.

First we recall the q-gamma function, cf. [3, Sect. 1.10],

$$\Gamma_q(z) = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1-q)^{1-z}, \qquad \lim_{q \uparrow 1} \Gamma_q(z) = \Gamma(z). \tag{4.1}$$

The q-gamma function can be used to see that (formally)

$$\lim_{q \uparrow 1} J_\nu(z(1-q)/2; q) = J_\nu(z). \tag{4.2}$$

See [13, Appendix A] for a rigorous limit result of this type.

In order to obtain Graf's addition formula (1.2) from (1.4) as $q \uparrow 1$ we have to consider the quotient of two Hahn–Exton q-Bessel functions, which we rewrite as an infinite sum of quotients of two Hahn–Exton q-Bessel functions of equal order. Explicitly,

$$\frac{J_x(q^{(1/2)(z-k)};q)}{J_{x-\nu}(q^{(1/2)z};q)}=q^{(1/2)\nu(z-k)}\sum_{m=0}^{\infty}\frac{(q^\nu;q)_m}{(q;q)_m}q^{m(1+(1/2)(x-\nu))}\times\frac{J_{x-\nu}(q^{(1/2)(z-k+m)};q)}{J_{x-\nu}(q^{(1/2)z};q)}. \tag{4.3}$$

Note that the Hahn–Exton q-Bessel functions on the right hand side of (4.3) only differ in the argument by a (half-)integer power of q. Equation (4.3) holds for $J_{x-\nu}(q^{(1/2)z};q)\neq 0$ and $\Re(x-\nu)>-1$. It can be proved by substituting the series representation (1.3) for the Hahn–Exton q-Bessel function $J_{x-\nu}$ in the nominator on the right-hand side of (4.3), interchanging summations, and using the q-binomial formula.

The limit $q\uparrow 1$ in (4.3) can be taken with the help of the following proposition. The proof of the proposition is an easy adaptation of the proof of [17, Theorem 1], which we will not give.

PROPOSITION 4.1. *Suppose* $\{p_k(x;n)\mid k\in\mathbf{Z}_+, n\in\mathbf{N}\}$ *is a series of functions satisfying a recurrence relation*

$$x^2p_k(x;n)=a_{k+1,n}p_{k+1}(x;n)+b_{k,n}p_k(x;n)+a_{k,n}p_{k-1}(x;n) \tag{4.4}$$

with $a_{k,n}>0$, $b_{k,n}\in\mathbf{R}$, *and initial conditions* $p_0(x;n)=f(x;n)$, $p_1(x;n)=g(x;n)$. *Assume that the zeros of* $p_k(x;n)$ *are real for all* $k\in\mathbf{Z}_+$ *and all* $n\in\mathbf{N}$ *and that*

$$\left|\frac{p_{k-1}(x;n)}{p_k(x;n)}\right|<\frac{C}{\delta},\qquad \forall k,n\in\mathbf{N}, \tag{4.5}$$

uniformly for $x\in K$ *for any compact subset* K *of* $\mathbf{C}\backslash\mathbf{R}$ *with* $d(K,\mathbf{R})>\delta$. *Moreover, assume that*

$$\lim_{n\to\infty}a_{n,n}=A>0,\qquad \lim_{n\to\infty}b_{n,n}=B\in\mathbf{R},$$

$$\lim_{n\to\infty}a_{k,n}^2-a_{k-1,n}^2=0,\qquad \lim_{n\to\infty}b_{k,n}-b_{k-1,n}=0,$$

uniformly in k.

Then we have

$$\lim_{n\to\infty}\frac{p_{n+1}(x;n)}{p_n(x;n)}=\rho\left(\frac{x^2-B}{2A}\right) \tag{4.6}$$

uniformly on compact subsets of $\mathbf{C}\backslash\mathbf{R}$ *with* $\rho(x)=x+\sqrt{x^2-1}$ *(and the square root is defined by* $|\rho(x)|>1$ *for* $x\notin[-1,1]$*).*

Next we apply Proposition 4.1 to

$$p_k(x;n)=(-1)^k\, q^{-(1/2)k} J_{2n\alpha+\beta}(xq^{-(1/2)k};q) \tag{4.7}$$

with q replaced by $c^{1/n}$ for a fixed $c\in(0,1)$. We assume $\alpha>0$ and $\beta>-1$, so that all zeros of $p_k(x;n)$ are real by [10, Corollary 3.2]. It follows from the second order q-difference equation for the Hahn–Exton q-Bessel function, cf. [15, (17), (18)], that (4.4) is satisfied with

$$a_{k,n}=q^{k-(1/2)+n\alpha+(1/2)\beta}, \qquad b_{k,n}=q^k(1+q^{2n\alpha+\beta}),$$

and $q=c^{1/n}$. Then the conditions on the coefficients $a_{k,n}$ and $b_{k,n}$ of Proposition 4.1 are easily verified. Moreover, $A=c^{1+\alpha}>0$ and $B=c(1+c^{2\alpha})\in\mathbf{R}$. It remains to prove that (4.5) holds.

LEMMA 4.2. *With* $p_k(x;n)$ *defined by* (4.7) *for* $\alpha>0$ *and* $\beta>-1$, *the estimate* (4.5) *holds.*

Proof. Consider the Wall polynomials for $0<b<1$,

$$w_p(x;b;q)=(-1)^p\sqrt{\frac{(b;q)_p}{b^p(q;q)_p}}\,{}_2\varphi_1\left(\begin{matrix}q^{-p},0\\ b\end{matrix};q,x\right).$$

From [17, (2.5) and Corollary 1] we get for $p\in\mathbf{N}$

$$\left|\frac{w_{p-1}(x;b;q)}{w_p(x;b;q)}\right|\leqslant\frac{q^p\sqrt{b(1-q^p)(1-bq^{p-1})}}{\delta'}\leqslant\frac{q^p\sqrt{b}}{\delta'}$$

for all $x\in K'$, K' compact in $\mathbf{C}\backslash[0,1]$, $\delta'=d(K',[0,1])$. Replace b, x by $q^{\mu+1}$, y^2q^{p-m} and take the limit $p\to\infty$. The Wall polynomials tend to the Hahn–Exton q-Bessel function uniformly, cf. [13, Proposition A.1], and we get

$$\left|\frac{J_\mu(yq^{-(1/2)m};q)}{J_\mu(yq^{-(1/2)(m+1)};q)}\right|\leqslant\frac{q^{m+(1/2)\mu}}{\delta}, \qquad \mu>-1 \tag{4.8}$$

for all $y\in K$, K a compact set of $\mathbf{C}\backslash\mathbf{R}$ with $\delta=d(K,\mathbf{R})$. The required estimate (4.5) for $p_k(x;n)$ defined by (4.7) follows from (4.8). ∎

Consequently, by Proposition 4.1,

$$\lim_{n\to\infty}\frac{J_{2n\alpha+\beta}(xc^{-1/2n}c^{-1/2};c^{1/n})}{J_{2n\alpha+\beta}(xc^{-1/2};c^{1/n})}=\rho=\rho(x;\alpha,c), \qquad \rho+\frac{1}{\rho}=\frac{c(1+c^{2\alpha})-x^2}{c^{1+\alpha}}. \tag{4.9}$$

In (4.3) we replace x, z, q by $2n\alpha$, $2n\gamma$, $c^{1/n}$, and we take $n \to \infty$. Iterating (4.9) and interchanging limit and summation shows that (4.3) tends to

$$c^{\nu\gamma} \sum_{m=0}^{\infty} \frac{(\nu)_m}{m!} c^{m\alpha} \rho^{k-m} = c^{\nu\gamma} \rho^k (1 - c^\alpha \rho^{-1})^{-\nu} \tag{4.10}$$

with $\rho + 1/\rho = (1 + c^{2\alpha} - c^{2\gamma})/c^\alpha$, $\nu < 1$, and $c^\gamma \in \mathbf{C}\backslash\mathbf{R}$. Interchanging limit and summation can be justified using the estimate (4.8), [11, Lemma A.1], and dominated convergence.

Finally, divide both sides of the addition formula (1.4) by $J_{x-\nu}(q^{(1/2)z}; q)$. Replace x, z, q by $2n\alpha$, $2n\gamma$, $c^{1/n}$ as before and y, R by $2n\eta$, $R(1-q)/2$. Take $n \to \infty$ and use (4.2), (4.10) to see that (1.4) formally tends to

$$\sum_{k=-\infty}^{\infty} \rho^k J_k(Rc^{\alpha+\eta})\, J_{\nu+k}(Rc^\eta) = c^{-\nu\gamma}(1 - \rho^{-1} c^\alpha)^\nu J_\nu(Rc^{\eta+\gamma}),$$

which is easily rewritten as Graf's addition formula (1.2) for the Bessel function.

5. The Limit Case $q \uparrow 1$ of the Product Formula

A (formal) limit transition of the product formula (1.5) for the Hahn–Exton q-Bessel function to the product formula (1.6) for the Bessel function is presented in this section.

We first rewrite the product formula (1.5). Use $J_m(z; q) = (-1)^m q^{-(1/2)m} J_{-m}(zq^{-(1/2)m}; q)$, replace m by $-m$, and use the formula

$$J_{x-\nu}(z; q) = z^{-\nu} \sum_{k=0}^{\infty} \frac{(q^{-\nu}; q)_k}{(q; q)_k} q^{k(1+(1/2)x)} J_x(zq^{(1/2)k}; q)$$

for $\Re(x) > -1$, and the notation $\int_0^\infty f(z)\, dm_q(z) = \sum_{z=-\infty}^{\infty} q^z f(q^{(1/2)z})$ to write (1.5) as

$$\begin{aligned} q^m J_{\nu+m}(Rq^{(1/2)(y+\nu+m)}; q)\, J_m(Rq^{(1/2)(x+y+m)}; q) \\ = \sum_{k=0}^{\infty} \frac{(q^{-\nu}; q)_k}{(q; q)_k} q^{k(1+(1/2)x)} \\ \times \int_0^\infty z^{-\nu} J_\nu(zRq^{(1/2)(y+\nu)}; q)\, J_x(zq^{-(1/2)m}; q)\, J_x(zq^{(1/2)k}; q)\, dm_q(z). \end{aligned} \tag{5.1}$$

In order to calculate the limit of the integral on the right-hand side of (5.1) we assume $x = 2n\alpha$, $q = c^{1/n}$, for $\alpha > 0$, $c \in (0, 1)$, so that we can use (4.7) (with $\beta = 0$) to find for $r \in \mathbf{Z}_+$,

$$\int_0^\infty z^{2r} J_x(zq^{-(1/2)m}; q)\, J_x(zq^{(1/2)k}; q)\, dm_q(z)$$
$$= (-1)^{m+k}\, c^{(1/2n)(m-k)} c^{-r} c^{-(k/n)r}$$
$$\times \int_0^\infty z^{2r} p_{n+m+k}(z; n)\, p_n(z; n)\, dm_{c^{1/n}}(z), \tag{5.2}$$

by shifting the summation parameter. The orthogonality relations (3.1) imply

$$\int_0^\infty p_k(z; n)\, p_l(z; n)\, dm_{c^{1/n}}(z) = \delta_{k,l}.$$

Repeating the first part of the Proof of Theorem 2 of Van Assche and Koornwinder [17] results in

$$\lim_{n \to \infty} \int_0^\infty z^{2r} p_n(z; n)\, p_{k+n}(z; n)\, dm_{c^{1/n}}(z)$$
$$= \frac{1}{\pi} \int_{B-2A}^{B+2A} \frac{z^r T_k((z-B)/2A)}{\sqrt{4A^2 - (z-B)^2}}\, dz \tag{5.3}$$

for $r, k \in \mathbf{Z}_+$, $A = c^{1+\alpha}$, $B = c(1 + c^{2\alpha})$ (as in Section 4) and $T_k(\cos\theta) = \cos k\theta$ is the Chebyshev polynomial of the first kind of degree k.

In (5.1) we replace R, x, y, and q by $\frac{1}{2}R(1-q)$, $2n\alpha$, $2n\eta$, $c^{1/n}$, and we use (4.2), (5.3), (5.2) to see that (5.1) (formally) tends to

$$J_{\nu+m}(Rc^\eta)\, J_m(Rc^{\alpha+\eta})$$
$$= \sum_{k=0}^\infty \frac{(-\nu)_k}{\pi k!} c^{\alpha k} (-1)^{k+m}$$
$$\times \int_{B-2A}^{B+2A} \left(\frac{z}{c}\right)^{-(1/2)\nu} J_\nu\left(Rc^\eta \sqrt{\frac{z}{c}}\right) \frac{T_{k+m}((z-B)/2A)}{\sqrt{4A^2 - (z-B)^2}}\, dz \tag{5.4}$$

as $n \to \infty$. In the integral on the right hand side of (5.4) we replace $(z-B)/2A$ by $\cos\psi$, so that the right-hand side of (5.4) equals

$$\frac{1}{2\pi} \sum_{k=0}^\infty \frac{(-\nu)_k}{k!} c^{\alpha k} (-1)^{m+k} \int_{-\pi}^{\pi} (2c^\alpha \cos\psi + 1 + c^{2\alpha})^{-(1/2)\nu}$$
$$\times J_\nu(Rc^\eta \sqrt{2c^\alpha \cos\psi + 1 + c^{2\alpha}})\, e^{-i(k+m)\psi}\, d\psi. \tag{5.5}$$

Replace ψ by $\psi+\pi$ in (5.5) and use the binomial theorem to see that (5.5) equals

$$\frac{1}{2\pi}\int_0^{2\pi} J_\nu(Rc^\eta\sqrt{1+c^{2\alpha}-2c^\alpha\cos\psi})\frac{(1-c^\alpha e^{-i\psi})^\nu}{(1+c^{2\alpha}-2c^\alpha\cos\psi)^{(1/2)\nu}}e^{-im\psi}\,d\psi. \tag{5.6}$$

Equating the left-hand side of (5.4) and (5.6) yields a formula equivalent to the product formula (1.6).

APPENDIX: JUSTIFICATIONS

In the Appendix we investigate the absolute convergence of two double sums occurring in Sections 2 and 3. The estimates used are based on the estimates given in [13]. In particular, cf. [13, (2.4)],

$$|J_\nu(z;q)|\leqslant|z^\nu|\frac{(-q^{\Re(\nu)+1},-q|z^2|;q)_\infty}{(q;q)_\infty}. \tag{A.1}$$

For integer order the Hahn–Exton q-Bessel function satisfies the estimate

$$|J_n(z;q)|\leqslant|z|^{|n|}\frac{(-q,-q|z^2|;q)_\infty}{(q;q)_\infty}\begin{cases}1, & n\geqslant 0;\\ q^{(1/2)n(n-1)}, & n\leqslant 0,\end{cases} \tag{A.2}$$

by [13, (2.4) and (2.6)]. A similar estimate also holds for the Hahn–Exton q-Bessel function with integer or half-integer power of q as argument because of the symmetry, cf. [13, (2.3)],

$$J_\alpha(q^{(1/2)\nu};q)=J_\nu(q^{(1/2)\alpha};q). \tag{A.3}$$

PROPOSITION A.1. *The double sum*

$$\sum_{n=0}^\infty\frac{(-1)^n q^{(1/2)n(n+1)}R^{2n}q^{n(x+\nu)}}{(q^{\nu+1};q)_n(q;q)_n}\times\sum_{z=-\infty}^\infty q^{z(n+1+(1/2)\nu)}J_x(q^{(1/2)(m+z)};q)\,J_{x-\nu}(q^{(1/2)z};q)$$

is absolutely convergent for $m\in\mathbf{Z}$, $\Re(x)>-1$, $|R|^2q^{1+\Re(x)+\Re(\nu)}<1$.

Proof. Assume $m\geqslant 0$ and use (A.2) and (A.3) to estimate

$$\sum_{z=0}^\infty|q^{z(n+1+(1/2)\nu)}J_x(q^{(1/2)(m+z)};q)\,J_{x-\nu}(q^{(1/2)z};q)|\leqslant C\sum_{z=0}^\infty q^{z(1+n+\Re(x))}\leqslant C$$

for $\Re(x) > -1$, $\forall n \in \mathbf{Z}_+$, and C is a constant independent of n of which the values may differ with each occurence. Consequently, the part $\sum_{n=0}^{\infty} \sum_{z=0}^{\infty}$ is absolutely convergent for $\Re(x) > -1$.

Similarly, we estimate

$$\sum_{z=-\infty}^{-m} |q^{z(n+1+(1/2)\nu)} J_x(q^{(1/2)(m+z)}; q)\, J_{x-\nu}(q^{(1/2)z}; q)|$$
$$\leqslant C \sum_{z=m}^{\infty} q^{z(1-n-m+\Re(x-\nu))} q^{z(z-1)}. \tag{A.4}$$

Shift the summation parameter to run from 0 to ∞, estimate $q^{z(z-1)}$ by $q^{(1/2)z(z-1)}/(q;q)_z$ and use the ${}_0\varphi_0$ summation formula to see that (A.4) can be estimated by

$$Cq^{-nm}(-q^{\Re(x-\nu)-n+1+m}; q)_n \leqslant Cq^{-(1/2)n(n-1)} q^{n\Re(x-\nu)}.$$

So the part $\sum_{n=0}^{\infty} \sum_{z=-\infty}^{-m}$ is absolutely convergent for $|R|^2 q^{1+\Re(x)+\Re(y)} < 1$.

The remaining sum over z is finite and causes no problems. The case $m \leqslant 0$ is essentially the same. ∎

PROPOSITION A.2. *The double sum*

$$\sum_{m=-\infty}^{\infty} q^m J_x(q^{(1/2)(l+m)}; q)$$
$$\times \sum_{z=-\infty}^{\infty} q^z J_x(q^{(1/2)(m+z)}; q)\, J_{x-\nu}(q^{(1/2)z}; q)\, J_\nu(Rq^{(1/2)(y+\nu+z)}; q)$$

is absolutely convergent for $\Re(x) > -1$, $|R|^2 q^{1+\Re(x)+\Re(y)} < 1$.

Proof. First we estimate the inner sum over z as a function of m. For the Hahn–Exton q-Bessel function of order x and $x-\nu$ we use the same estimates as before, and from (A.1) we get

$$|J_\nu(Rq^{(1/2)(y+\nu+z)}; q)|$$
$$\leqslant Cq^{(1/2)\Re(\nu)z} \begin{cases} 1, & z \geqslant 0; \\ |R|^{-2z} q^{-z(\Re(y+\nu)+1)} q^{-(1/2)z(z-1)}, & z \leqslant 0, \end{cases}$$

where C is independent of m. Denote by $S_m(1)$ the inner sum over z from $\max(0, -m)$ to ∞, then we find, using these estimates, for $\Re(x) > -1$,

$$|S_m(1)| \leqslant C \begin{cases} q^{(1/2)m\Re(x)}, & m \geqslant 0; \\ q^{-m-(1/2)m\Re(x)}, & m \leqslant 0. \end{cases}$$

For $S_m(2)$, the inner sum over z from $-\infty$ up to $\min(0, -m)$, we obtain

$$|S_m(2)| \leqslant C \begin{cases} q^{m((1/2)\Re(x)+\Re(y))}\,|R|^{2m}, & m \geqslant 0; \\ q^{-(1/2)m\Re(x)+(1/2)m(m-1)}, & m \leqslant 0. \end{cases}$$

The remaining finite sum over z from $\min(0, -m)+1$ to $\max(0, -m)-1$, denoted by $S_m(3)$, can be estimated by

$$|S_m(3)| \leqslant C \begin{cases} q^{(1/2)m\Re(x)}(1-(|R|^2\,q^{\Re(y)})^{m-1}), & m \geqslant 0; \\ q^{-(1/2)m\Re(x)}, & m \leqslant 0. \end{cases}$$

To estimate the double sum of the proposition we combine (A.2) and (A.3) to obtain an estimate on $J_x(q^{(1/2)(l+m)}; q)$. The sum $m=-\infty$ to $\min(0, -l)$ and the finite sum from $m=\min(0, -l)+1$ to $\max(0, -l)-1$ are absolutely convergent. The sum $m=\max(0, -l)$ to ∞ is absolutely convergent for $\Re(x) > -1$ and $q^{1+\Re(x)+\Re(y)}\,|R|^2 < 1$. ∎

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